Quantum discord and its geometric measure with death entanglement in correlated dephasing two qubits system

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Abstract. The quantum correlations, including entanglement and discord with its geometric measure, and classical correlation are studied for a bipartite partition of a open or closed quantum system. It is found that the purity of the initial state plays an important role in the dynamics of quantum and classical correlations. In the dephasing model, the quantum correlations loss and the classical correlation gain are instantaneously happen. While, the purity of the initial state destroys the quantum correlations which is resulted by the unitary interaction. Therefore, with the purity parameter, a particular region in which there is no state have quantum correlations can be determined.

Keywords:Quantum discord; geometric measure of quantum discord; dephasing model

1. Introduction

The development of the quantum information technology stimulates a deep study of the properties of quantum correlations inherent in a quantum system. In particular, there is a problem of identification of those quantum correlations which are responsible for the advantages of the quantum computations in comparison with the classical ones. Therefore, lots of interest have been devoted to the definition and understanding of correlations in quantum systems in the last two decades. Specially, the definition and study of quantum and classical correlations in quantum systems. It is well known that the total correlation in a bipartite quantum system can be measured by quantum mutual information [1], which may be divided into classical and quantum parts [2-5]. The quantum part is called quantum discord(QD) which is originally introduced by [4]. Recently, it has been aware of the fact that quantum discord is a more general concept to measure quantum correlation than quantum entanglement(QE) since there is a nonzero quantum discord in some separable mixed states [4]. The dynamics of quantum discord and entanglement has been recently compared under the same conditions when entanglement dynamic undergoes a sudden death [6, 7, 8, 9, 10]. It was shown that quantum discord presents an instantaneous disappearance at some time points in non-Markovian regime [6], and exponential decay in Markovian regime[7]. Interestingly,

it has been proven both theoretically and experimentally that such states provide computational speedup compared to classical states in some quantum computation models[11, 12]. In these contexts, quantum discord could be a new resource for quantum computation.

The calculation of quantum discord is based on numerical maximization procedure, it does not guarantee exact results and there are few analytical expressions including special cases [13, 14]. To avoid this difficulty, geometric measure of quantum discord (GMQD) is introduced by Ref.[15], which measures the quantum correlations through the minimum Hilbert-Schmidt distance between the given state and zero discord state.

Because of the unavoidable interaction between a quantum system and its environment, understanding the dynamics of quantum and classical correlations (CC) is an interesting line of research [14-17]. The influence of Markovian [7] and non-Markovian [17] environment on the dynamics of QD and GMQD. They showed that both GMQD and QD die asymptotically with entanglement sudden death, and the discontinuity in the decay rate of GMQD does not always imply the discontinuity in the decay rate of QD. Also, they observed that even when QD vanishes at discrete times, GMQD disappears but not instantly. But in our work the dephasing environment, in which energy transfer from the system to the environment does not occur, is considered. Some work has been devoted to this issue [20, 21, 22], in which, the authors show that disentanglement is dependent on the initial condition and temperature of the environment. This type of the environment leads to phenomenon entanglement sudden death, i.e., the entanglement can decrease to zero abruptly and remains zero for a finite time [23, 24, 25]. Entanglement sudden death has been experimentally observed in an implementation using twin photons [26], and atomic ensembles [27].

In this paper, one considers a dephasing model, in which two qubits embed into a multi-mode quantized field and the interaction between the two qubits is also considered. Therefore, the quantum correlations via quantum discord and its geometric measure (GMQD) is compared with both quantum entanglement and classical correlation.

2. Measures of correlations

2.1. Quantum discord

To quantify the quantum correlations of a bipartite system, no matter whether it is separable or entangled, one can use the quantum discord [2, 4]. Quantum discord measures all nonclassical correlations and defined as the difference between total correlation and the classical correlation with the following expression

$$D(\rho^{AB}) = \mathcal{I}(\rho^{AB}) - \mathcal{Q}(\rho^{AB}), \qquad (1)$$

which quantifies the quantum correlations in ρ^{AB} and can be further distributed into entanglement and quantum dissonance (quantum correlations excluding entanglement)[28]. Here the total correlation between two subsystems A and B of a

bipartite quantum system ρ^{AB} is measured by quantum mutual information,

$$\mathcal{I}(\rho^{AB}) = \mathcal{S}(\rho^A) + \mathcal{S}(\rho^B) - \mathcal{S}(\rho^{AB}), \tag{2}$$

where $\mathcal{S}(\rho^{AB}) = Tr(\rho^{AB} \log \rho^{AB})$ is the Von Neumann entropy, $\rho^A = Tr_B(\rho^{AB})$ and $\rho^B = Tr_A(\rho^{AB})$ are the reduced density operators of the subsystems A and B, respectively. The measure of classical correlation is introduced implicitly in Ref.[4] and interpreted explicitly in the Ref.[2]. The classical correlation between the two subsystems A and B is given by

$$\mathcal{Q}(\rho^{AB}) = \max_{\{\Pi_k\}} [\mathcal{S}(\rho^A) - \sum_k p_k \mathcal{S}(\rho_k)], \tag{3}$$

where $\{\Pi_k\}$ is a complete set of projectors to measure the subsystem B, and $\rho_k = Tr_B[(I^A \otimes \Pi_k)\rho_{AB}(I^A \otimes \Pi_k)]/p_k$ is the state of the subsystem A after the measurement resulting in outcome k with the probability $p_k = Tr_{AB}[(I^A \otimes \Pi_k)\rho^{AB}(I^A \otimes \Pi_k)]$, and I^A denotes the identity operator for the subsystem A. Here, maximizing the quantity represents the most gained information about the system A as a result of the perfect measurement $\{\Pi_k\}$. It can be shown that quantum discord is zero for states with only classical correlations and nonzero for states with quantum correlations. Note that discord is not a symmetric quantity, i.e., its amount depends on the measurement performed on the subsytem A or B [15].

2.2. Geometric measure of quantum discord

The geometric measure of quantum discord quantifies the quantum correlation through the nearest Hilbert-Schmidt distance between the given state and the zero discord state [15, 16], which is given by

$$D_A^g = \min_{\chi \in S} \|\rho^{AB} - \chi\|^2, \tag{4}$$

where S denotes the set of zero discord states and $\|A\|^2 = Tr(A^\dagger A)$ is the square of Hilbert-Schmidt norm of Hermitian operators. The subscript A of D_A^g implies that the measurement is taken on the system A. A state χ on $H^A \otimes H^B$ is of zero discord if and only if it is a classical-quantum state [29], which can be represented as

$$\chi = \sum_{k=1}^{2} p_k |k\rangle\langle k| \otimes \rho_k,$$

where $\{p_k\}$ is a probability distribution, $|k\rangle$ is an arbitrary orthonormal basis for H^A and ρ_k is a set of arbitrary states (density operators) on H^B. An easily computable exact expression for the geometric measure of quantum discord is obtained by Ref.[15] for a two qubit system, which can be described as follows. Consider a two-qubit state ρ^{AB} expressed in its Bloch representation as

$$\rho^{AB} = \frac{1}{4}[I^A \otimes I^B + \sum_{i=1} x_i(\sigma_i \otimes I^B + I^A \otimes y_i\sigma_i)$$
$$+ \sum_{ij=1} R_{ij}\sigma_i \otimes \sigma_j], \tag{5}$$

where $\{\sigma_i\}$ are the usual Pauli spin matrices. The components of the local Bloch vector are $x_i = Tr(\rho^{AB}(\sigma_i \otimes I))$ and $y_i = Tr(\rho^{AB}(I \otimes \sigma_i))$. $R_{ij} = Tr(\rho^{AB}(\sigma_i \otimes \sigma_j))$ are the components of the correlation matrix[15]. Therefore, its geometric measure of quantum discord is given by

$$D_A^g = \frac{1}{4}(\|\vec{x}\|^2 + \|y\|^2 - k_{max}),\tag{6}$$

$\vec{x} = (x_1, x_2, x_3)^T$, R is the matrix with elements R_{ij} and k_{max} is the largest eigenvalue of the matrix $K = \vec{x}\vec{x}^T + RR^T$.

2.3. Entanglement via Negativity

Here, one uses the negativity[30] to measure the entanglement, i.e., the negative eigenvalues of the partial transposition of ρ^{AB} are used to measure the entanglement of the qubits system. Therefore, the negativity of a state ρ^{AB} is defined as

$$N(\rho) = \max(0, -2\sum_j \mu_j),\tag{7}$$

where μ_j is the negative eigenvalue of $(\rho^{AB}(t))^{T_B}$, and T_B denotes the partial transpose with respect to the second system.

3. Quantum and classical correlations in two-qubit models

In this section one tries to present, by examining two examples, an physical interpretation for the relation between quantum entanglement, quantum and classical correlations. The first model consists of two non-interaction qubits couple with the same quantized field under the rotating-wave approximation. Another is the dephasing model, in which two qubits embed into a multi-mode quantized field and the interaction between the two qubits is also considered.

3.1. Two non-interaction qubits couple with the field

Here, one considers two qubits coupled with a single-mode cavity field, which is in the case of the exact resonance with the qubits. One of the pioneering potential applications on this Hamiltonian in the context of quantum information is " Cooper pair box"(qubits)[31], i.e., the coupled system of two Cooper pair box (artificial atoms) and photons stored in the resonator (cavity mode). The cavity is sustaining non-decaying single mode field in its thermal state along with the mode structure of the electromagnetic field. The interaction picture Hamiltonian with the rotating wave approximation is given by

$$\hat{H} = \lambda \sum_{k=1}^{2} (\hat{a}^\dagger |0\rangle_k \langle 1| + \hat{a}|1\rangle_k \langle 0|),\tag{8}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for the cavity mode, $|0\rangle_k$ and $|1\rangle_k$ denote to the ground and excited states of the k-th qubit, respectively. For the

whole system (system+field), the evolution of the whole system is characterized by the interaction between the two-qubits system and single-mode cavity. However from the point of the qubits, the energy transfer between the qubits and the field happens, which is described by the relaxation term $\hat{a}^{\dagger}|0\rangle_k\langle 1|$ and the backaction term $\hat{a}|1\rangle_k\langle 0|$. One assumes that the two qubits are initially prepared in Werner states, which is defined by

$$\rho^{AB}(0) = p|\varphi\rangle\langle\varphi| + \frac{1}{4}(1-p)\hat{I}, \tag{9}$$

where $|\varphi\rangle = \sin\theta|11\rangle + \cos\theta|00\rangle$ and p is a real number which indicates the purity of initial state, $\hat{I}$ is a 4×4 identity matrix. But the cavity field is initially prepared in the vacuum state, i.e., $\rho^F(0) = |0\rangle\langle 0|$. Then the initial density operation for the whole qubits-field system is: $\rho^{ABF}(0) = \rho^{AB}(0) \otimes |0\rangle\langle 0|$. By using the above initial states, the density matrix of the qubits-field system with the interaction (8) evolves to $\rho(t) = \hat{U}(t)\rho^{ABF}(0)\hat{U}^{\dagger}(t)$, where the time evolution operator $\hat{U}(t) = \exp(-i\hat{H}t)$. The reduced density matrix, $\rho^{AB}(t)$, of two qubits is calculated by tracing out the cavity field variables. Therefore, $\rho^{AB}(t)$ is given by

$$\begin{aligned}
\rho^{AB}(t) = {} & a_1|11\rangle\langle 11| + a_2|00\rangle\langle 00| + a_3(|11\rangle\langle 00| \\
& + |00\rangle\langle 11|) + a_4(|10\rangle\langle 10| + |01\rangle\langle 01|) \\
& + a_5(|10\rangle\langle 01| + |01\rangle\langle 10|)
\end{aligned} \tag{10}$$

with the abbreviation

$$\begin{aligned}
a_1 &= \frac{1-p}{4} + \frac{p}{9}(2 + \cos\varpi t)^2 \sin^2\theta, \\
a_2 &= \frac{1-p}{4} + p\cos^2\theta + \frac{2p}{9}(1 - \cos\varpi t)^2 \sin^2\theta, \\
a_3 &= \frac{p}{3}(2 + \cos\varpi t)\sin\theta\cos\theta, \\
a_4 &= \frac{1-p}{4} + \frac{p}{6}\sin^2\varpi t \sin^2\theta, \qquad a_5 = a_4 - \frac{1-p}{4},
\end{aligned}$$

where $\varpi = 2.4495\lambda$. The eigenvalues of the density matrix $\rho^{AB}(t)$ are given by: $\lambda_{1,2} = a_4 \pm a_5$, and $\lambda_{3,4} = \frac{1}{2}[(a_1 + a_2) \pm \sqrt{(a_1 - a_2)^2 - 4a_3^2}]$. After some straightforward calculation, the reduced density matrices associated with the above states is given by

$$\begin{aligned}
\rho^A(t) &= \rho^B(t) \\
&= (a_1 + a_4)|11\rangle\langle 11| + (a_2 + a_4)|00\rangle\langle 00|.
\end{aligned} \tag{11}$$

One noted that, the reduced density matrices of the qubits are represented in diagonal matrices. Therefore, these the states $\rho^A(t)$ and $\rho^B(t)$ are classical states.

For the case of the two qubits coupled with a single-mode cavity field, the results are given in Figs.1a,b. In these figures, one reports the dynamics of D_A^g, $D(\rho^{AB})$, $N(\rho)$ and $Q(\rho)$ as a functions of the time λt for different values of the purity of initial state (namely $p = 0.5, 1.0$) with $\theta = \frac{\pi}{4}$. It is worth noting to mention that, because the matrix of the initial state of the two qubits is not a diagonal matrix, this state is not classical state. Therefore, its quantum correlation have non zero value at $\lambda t = 0.0$. For $p = 1.0$, these measures instantaneously oscillate and reach their maximum (at
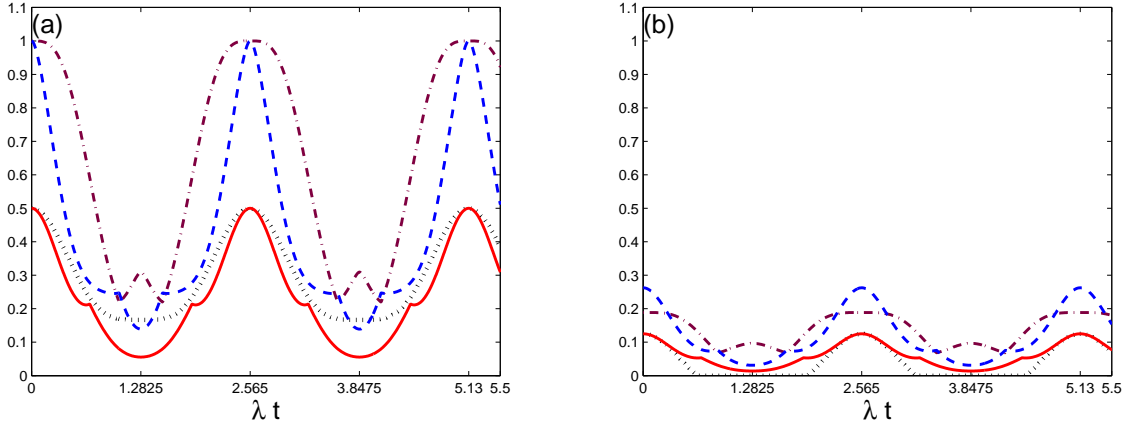
Figure 1. Time evolutions of the quantum discord (dash plots), the geometric measure of QD (sold plots), the negativity (dotted plots) and the classical correlation (dash-dot plots) for $p = 1$ in (a) and $p = 0.5$ in (b) for $\theta = \frac{\pi}{4}$.

$\lambda t = \frac{2n\pi}{\omega}, n = 0, 1, 2, ...$) and minimum values (at $\lambda t = \frac{(2n+1)\pi}{\omega}$) at the same time points (see Fig.1a). Because the function $\cos \varpi t$ is a periodical function on the scaled time with period $\frac{2\pi}{\varpi}$, D_A^g, $D(\rho)$, $N(\rho)$ and $Q(\rho)$ evolve periodically with respect to the scaled time with period $\frac{2\pi}{\varpi}$ (see Figs.1a,b). A rather counterintuitive feature of the QE is that it may exceed the measures of the GMQD and QD (see Fig.1a). So one can say that GMQD and QD are more general than QE.

For $p = 0.5$ (see Fig.1b), one can observe that the phenomenon of entanglement death occurs, but this phenomenon does not occur for GMQD and QD even when the purity p is small. Because the entanglement undergoes sudden death while the correlations are long lived, QE is not greater than QD, GMQD and CC for some time. Also one can see that one value of QE corresponds to many values of QD and GMQD, meaning that the states in possession of the same entanglement give different correlations. This means that, there are correlations(quantum and classical) in the intervals of entanglement death. Therefore, the entanglement is not the only part of quantum correlations. This agrees with Ref.[29], which showed that *absence of entanglement does not imply classicality*. From Fig.1a, one can see that the intervals of vanishing negativity disappear when $p = 1$. Therefore, the entanglement sudden death is completely sensitive to the purity of the initial-state.

3.2. Dephasing two interaction qubits by a multimode quantized field

Here, one considers a dephasing model of two qubits embed into a multimode quantized field and the interaction between the two qubits is also considered. The dephasing channel case is important situation for the open systems, in which there is no energy transfer between the system and environment. The Hamiltonian of this case can be
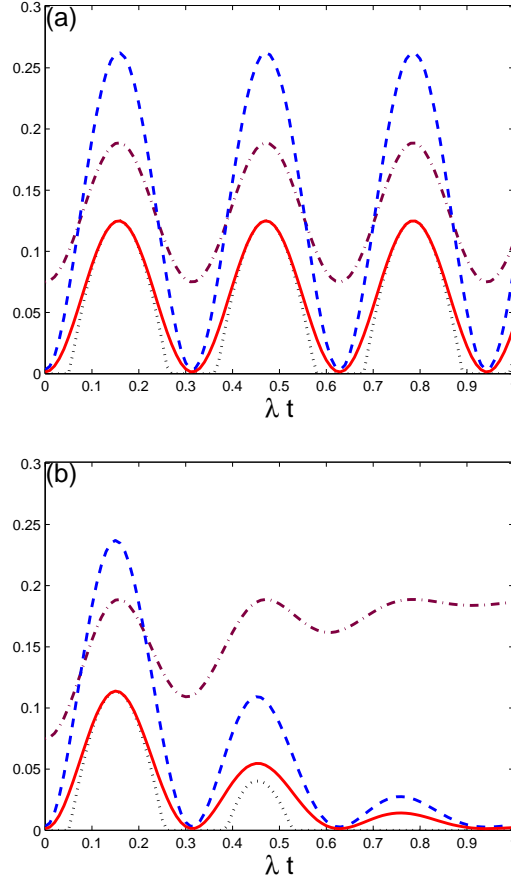
Figure 2. Time evolutions of the quantum discord (dash plots), the geometric measure of QD (sold plots), the negativity (dotted plots) and the classical correlation (dash-dot plots) for $\gamma = 0.0\lambda$ in (a) and $\gamma = 1.0\lambda$ in (b)for $\theta = \frac{\pi}{60}$ and $p = 0.5$.

written as[33]

$$
\hat{H} = \frac{\omega_0}{2}(\sigma_A^z + \sigma_B^z) + \lambda(\sigma_A^+ \sigma_B^- + \sigma_B^+ \sigma_A^-)
$$
$$
+ \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k + \gamma_k(\sigma_A^+ \sigma_B^- + \sigma_B^+ \sigma_A^-)(\hat{b}_k^\dagger + \hat{b}_k), \tag{12}
$$

where ω_o is the qubit transition frequency and ω_k the frequency of the k-th field, the coupling constant between the two qubits is λ but γ_k is system-reservoir coupling constant (dephasing channel parameter).

By using the same previous initial states but with $|\varphi\rangle = \sin\theta|10\rangle + \cos\theta|01\rangle$, the density matrix of the dephasing qubits-field system is given by: $\hat{U}(t)\rho^{ABF}(0)\hat{U}^\dagger(t)$. Therefore, the reduced density matrix of the two qubits is given by

$$
\rho^{AB}(t) = \alpha_1(|11\rangle\langle11| + |00\rangle\langle00|) + \alpha_2|10\rangle\langle10|
$$
$$
+ \alpha_3|01\rangle\langle01| + \alpha_4|10\rangle\langle01| + \alpha_4^*|01\rangle\langle10|, \tag{13}
$$

with the abbreviation

$$
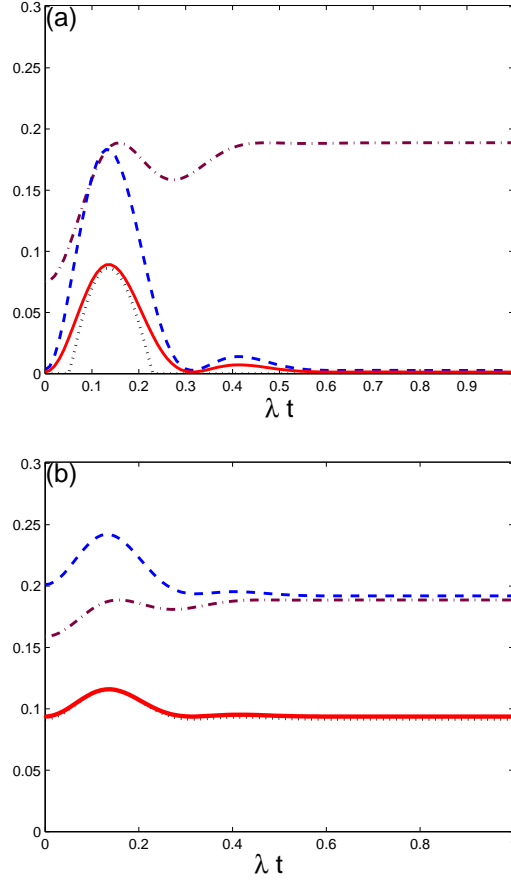\alpha_1 = (1-p)/4, \qquad\qquad \alpha_2 = (1+p)/4 - \beta,
$$

Figure 3. The same as in Fig.2 but for $(\frac{\gamma}{\lambda}, \theta) = (2, \frac{\pi}{60})$ in (a) and $(\frac{\gamma}{\lambda}, \theta) = (2, \frac{\pi}{3})$ in (b)

$$\alpha_3 = (1+p)/4 + \beta, \qquad \alpha_4 = \frac{p}{2} \sin 2\theta - i\beta,$$

$$\beta = \frac{p}{2} L_d \cos 2\theta \cos 2\lambda t,$$

$$L_d = \exp[-4 \sum_k (\frac{\gamma_k}{\omega_k})^2 (1 - \cos \omega_k)].$$

Where L_d is the decoherence factor which leads to damping of the off-diagonal terms. One can get the eigenvalues of the density matrix $\rho(t)$ as

$$\lambda_{1,2} = \alpha_1, \quad \lambda_{3,4} = \frac{1}{4}[1 + p \pm \sqrt{(1+p)^2 - 16|\alpha_4|^2}] \tag{14}$$

The reduced density matrices associated with the above states are given by

$$\rho^A(t) = (\alpha_1 + \alpha_2)|11\rangle\langle 11| + (\alpha_1 + \alpha_3)|00\rangle\langle 00|, \tag{15}$$

$$\rho^B(t) = (\alpha_1 + \alpha_3)|11\rangle\langle 11| + (\alpha_1 + \alpha_2)|00\rangle\langle 00|. \tag{16}$$

The results of the case of the dephasing model are given in Figs.2-**??**. In Fig.2, GMQD, QD, QE and CC are plotted as a functions of the time λt for different values of system-reservoir coupling parameterγ (namely $\gamma/\lambda = 0.0, 1, 2$) with $\theta = \frac{\pi}{60}$ and $p = 0.5$. From

Fig.2a, one can easily find the common features of the dynamics of GMQD, QD, QE and CC. The previous measures of quantum and classical correlations present instantaneous oscillations and reach their extreme values at the same time points. Because the entanglement undergoes sudden death for $\gamma \neq 0$, while D_A^g, $D(\rho)$ and $Q(\rho)$ are long lived. $N(\rho)$ is not greater than D_A^g, $D(\rho)$ and $Q(\rho)$ for small intervals and QE attains constant values during an intervals while the quantum correlations (GMQD and QD) vary in these intervals.

In Figs.2b,3a,b, one examines the effect of system-reservoir coupling γ on the dynamics of the previous measures of quantum and classical correlations with $\theta = \frac{\pi}{60}$. It is clear that D_A^g, $D(\rho)$ and $N(\rho)$ decrease with increasing of γ, and they have zero values. Precisely, the dephasing parameter γ leads to exponentially decay for maximum values of the D_A^g, $D(\rho)$ and $N(\rho)$ to zero value, while $Q(\rho)$ exponentially evolves to its asymptotic value. It is interesting to note that the larger the value of γ is, the more rapidly D_A^g, $D(\rho)$ and $N(\rho)$ reach its asymptotic values of zero. This means that, quantum correlations, including entanglement and discord with its geometric measure die asymptotically with large values of γ. On the contrary, the classical correlation increases with increasing γ, and it have nonzero values for $\gamma > 0$. The figures show that the classical correlation approaches an almost steady state for large values of γ faster than that for small values of γ. Finally, after a very long time, the classical correlation loses it oscillations and asymptotically reaches its steady state, i.e., the final state of the qubits reaches a classical state. One can say that, the quantum states with large γ are rapidly transformed into classical states, i.e., the processes of quantum correlation loss and classical correlation gain are instantaneously happen.

To investigate the influence of $\theta = \frac{\pi}{3}$ (mixed state) with a large value of $\gamma = 2\lambda$ on the correlations see Fig.3,b. From this figure, one notes that both QD and CC reach their asymptotic values and approximately have the same behavior while GMQD and QE have the same behavior. This shows that the mixedness of the initial states affects on all the previous measures in a similar way and it inhibits them from going into zero.   In Figs.**??,??**, one examines the effect of the purity of the initial states on the dynamics of the previous measures with $\theta = \frac{\pi}{3}$ and $\gamma = 0.8\lambda$. It is clear that they decrease with decreasing of the purity parameter p. When the purity p is zero, all the measures vanish, i.e, the mixedness of the initial states have the same effect on all measures. One sees that the influence of purity leads to: the amplitudes of the local maxima of D_A^g, $D(\rho)$, $N(\rho)$ and $Q(\rho)$ have exponential decay with decreasing the parameter p. When quantum correlations measures quite vanishes, the states ρ^{AB} finally go into a classical state and its quantum correlation is lost completely. This means that, after a particular time, the purity destroys the quantum correlations of the qubits which is resulted by the unitary interaction. Therefore, in the presence of purity one can determines a particular region, in which, there is no state have quantum correlations.

4. Conclusions

The dynamics of quantum correlations, including entanglement and discord with its geometric measure, and classical correlation in two-qubit models are introduced for a open or closed quantum system. It is found that the dynamics of GMQD, QD, QE and CC differ. Where, quantum discord and its geometric measure are exist in the region where the entanglement is zero, which is a strong signature for the presence of non classical correlations. System-reservoir coupling leads to: GMQD, QD and QE die asymptotically with larger system-reservoir coupling parameter. Also, processes of quantum correlation loss and classical correlation gain are instantaneously happen. It is found that the purity of the initial states destroys the quantum correlations by exponential decay. Therefore, in presence of the purity, one can determines a particular region in which there is no state have quantum correlations.

[1] B. Schumacher, M. D. Westmoreland, *Phys. Rev.* A **74**(2006) 042305; B. Groisman, S. Popescu, A. Winter, *Phys. Rev.* A **72** (2005) 032317.

[2] L. Henderson and V. Vedral, *J. Phys. A: Math. Gen.* **34** (2001) 6899; V. Vedral, *Phys. Rev. Lett.* **90** (2003) 050401.

[3] D. Yang, M. Horodecki, Z. D. Wang *Phys. Rev. Lett.* **101**(2008) 140501.

[4] H. Ollivier and W. H. Zurek, *Phys. Rev. Lett.* **88** (2001) 017901.

[5] M. Piani, P. Horodecki, R. Horodecki, *Phys. Rev. Lett.* **100** (2008)090502; M. Piani, M. Christandl, C. E. Mora, P. Horodecki, *Phys. Rev. Lett.* **102**(2009) 250503.

[6] B. Wang, Z.-Y. Xu, Z.-Q. Chen, M. Feng, *Phys. Rev.* A **81** (2010) 014101.

[7] T. Werlang, S. Souza, F.F. Fanchini, C. J. Villas Boas, *Phys. Rev.* A **80** (2009) 024103.

[8] Xing Xiao, Mao-Fa Fang, Yan-Ling Li, Guo-Dong Kang, Chao Wu, *Opt. Commun.* **283** (2010) 3001; T. Werlang, G. Rigolin, *Phys. Rev.* A **81** (2010) 044101.

[9] A.-B. A. Mohamed, *Int J Theor Phys.* **51**(2012) 2779.

[10] A.-B. A. Mohamed, *Phys. Scr.* **85** (2012) 055013.

[11] A. Datta, A. Shaji, C. M. Caves, *Phys. Rev. Lett.* **100**(2008) 050502.

[12] B.P. Lanyon, M. Barbieri, M. P. Almeida, A.G. White, *Phys. Rev. Lett.* **101** 200501 (2008).

[13] F. F. Fanchini, T. Werlang, C. A. Brasil, L. G. E. Arruda, and A. O. Caldeira, *Phys. Rev.* A **81** (2010) 052107; M.S. Sarandy, *Phys. Rev.* A **80** (2009) 022108.

[14] M. Ali, A. R. P. Rau, G. Alber, *Phys. Rev.* A **81** (2010) 042105; S. Luo, *Phys. Rev.* A **77** (2008) 042303.

[15] B. Dakic, V. Vedral, and C. Brukner, *Phys. Rev. Lett.* **105** (2010)190502.

[16] X. M. Lu, Z. J Xi, Z. Sun and X. Wang, *Quantum information and computation* **10** (2010)0994.

[17] F. Altintas, *Opt. Commun.* **283** (2010)5264.

[18] B. Wang, Z.-Y. Xu, Z.-Q. Chen, M. Feng, *Phys. Rev.* A **81** (2010) 014101.

[19] Jun-Qi Li, J.-Q. Liang, *Phys. Lett.* A **375** (2011) 1496.

[20] K. Roszak, P. Machnikowski, *Phys. Rev.* A **73** (2006) 022313.

[21] A.-B. A. Mohamed, *Ann. Phys.* **327** (2012) 3130; *Quantum Inf Process* DOI 10.1007/s11128-012-0460-1.

[22] Abdel-Baset A. Mohamed, *Journal of the Egyptian Mathematical Society*, DOI 10.1016/j.joems.2012.10.005.

[23] T. Yu, J.H. Eberly, *Phys. Rev. Lett.* **93** (2004) 140404.

[24] T. Yu, J.H. Eberly, *Phys. Rev. Lett.* **97** (2006) 140403; T. Yu, *Phys. Lett. A* (2007)361.

[25] J. H. Eberly, T. Yu, *Science* **316** (2007) 555; T. Yu, J.H. Eberly, *Science* **323** (2009) 598.

[26] M. P. Almeida, F. de Melo, M. Hor-Meyll, A. Salles, S. P. Walborn, P. H. Ribeiro Souto, L. Davidovich, *Science* **316** (2007) 579.

[27] J. Laurat, K. S. Choi, H. Deng, C. W. Chou, H. J. Kimble, *Phys. Rev. Lett.* **99** (2007) 180504.

[28] K. Modi, T. Paterek, W. Son, V. Vedral, M. Williamson, *Phys. Rev. Lett.* **104** (2010)080501.

[29] S. Luo, *Phys. Rev.* A **77** (2008) 022301; S. Luo and Q. Zhang, *J. Stat. Phys.* **131** (2008)1169.

[30] G. Vidal, R. F. Werner, *Phys. Rev.* A **65**(2002) 032314.

[31] A. Blais, R.-S. Huang, A. Wallraff, S. Girvin and R. Schoelkopf, *Phys. Rev.* A **69** (2004) 062320.

[32] R. F. Werner, *Phys. Rev.* A **40** (1989) 4277.

[33] C. P. Sun, X. X. Yi, X. J. Liu, *Fortschr. Phys.* **43**(1995) 585.

(b)